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Spreading Processes and Large Components in Ordered, Directed Random Graphs

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Abstract

Order the vertices of a directed random graph $v_1, \dots, v_n$; edge (v_i, v_j) for $i < j$ exists independently with probability p. This random graph model is related to certain spreading processes on networks. We consider the component reachable from v_1 and prove existence of a sharp threshold $p^* = \log n/n$ at which this reachable component transitions from $o(n)$ to $\Omega(n)$.

1 Introduction

In this note we study a random graph model that captures the dynamics of a particular type of spreading process. Consider a set of n ordered vertices $\{v_1, \dots, v_n\}$ with vertex v_1 initially 'infiltrated' (at time step 1). At time steps $2, 3, \dots, n$, vertex v_1 attempts to independently infiltrate, with probability p, each of $v_2, v_3 \dots, v_n$ in turn (one per step). Either v_i gets infiltrated or immunized. If v_i is infected, it attempts to infect $v_{i+1}, \dots, v_n$, also each with probability p; v_i does not attempt to infect $v_1, \dots, v_{i-1}$, however, as prior vertices are already either infiltrated or immunized. At time step i, all infiltrated vertices v_j with $j < i$ are attempting to infiltrate v_i, and v_i gets infiltrated if any one of these attempts succeeds. Intuitively, v_i is more likely to get infiltrated if more vertices are already infiltrated at the time that v_i becomes 'succeptible'. One example of such a contagion process is given in [6].

This spreading process is equivalent to the following random model of an ordered, directed graph G: order the vertices $v_1, \dots, v_n$, and for $i < j$, the directed edge (v_i, v_j) exists in G with probability p (independently). Vertex v_i is infected if there is a (directed) path from v_1 to v_i. The question we address is, "What is the size of the set of vertices reachable from v_1?" (the size of the infection). We prove the following sharp result.

Theorem 1. *Let $\mathcal{R}$ be the set of vertices reachable from v_1, and suppose $p = \frac{c \log n}{n} + \xi(n)$, where $\xi(n) = o(\frac{\log n}{n})$ and $c > 0$ is fixed. Then:*

1. *If $c < 1$, then $|\mathcal{R}| = n^{c+o(1)}$, a.a.s.*

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2. *If* $c = 1$, *then* $|\mathcal{R}| = o(n)$, *a.a.s.*

3. *If* $c > 1$, *then* $|\mathcal{R}| = \left(1 - \frac{1}{c} + o(1)\right) n$, *a.a.s.*

Recall that an event holds a.a.s.(asymptotically almost surely), if it holds with probability $1 - o(1)$; that is it holds with probability tending to one as n tends to infinity. Note that we do not explicitly care whether $\xi(n)$ is positive or negative in the results above.

Similar phase transitions are well known for various graph properties in other random graph models. As shown by Erdős and Rényi in [2], in the $G(n, M)$ model of random graphs, where a graph is chosen independently from all graphs with M edges, there is a similar emergence of a component of size $\Theta(n)$ around $M = \frac{n}{2}$ edges. Likewise, a threshold for connectivity was shown for $M = \frac{n \log n}{2}$ edges. For the more familiar $G(n, p)$ model, where edges are present independenty with probability p, this translates into a threshold at $p = \frac{1}{n}$ for a giant component, and at $p = \frac{\log n}{n}$ for connectivity. A much more comprehensive account of results on properties of random graphs can be found in [1]. Łuczak in [4] and more recently Łuczak and Seierstad in [5], studied the emergence of the giant component in a random directed graphs, in both the directed model where M random edges are present and in the model where edges are present with probability p. Thresholds for strong connectivity were established for random directed graphs by Palásti [7] (for random directed graphs with M edges) and Graham and Pike [3] (for random directed graphs with edge probability p). We are not aware of any results for ordered directed random graphs where edges connect vertices of lower index to higher index.

2 A Proof of Theorem 1

Upper bounds: For $i > 1$, let $\mathcal{R}_i$ denote the event that v_i is reachable, and let X_i denote the number of paths to vertex v_i in G. If $\mathcal{P}_i$ denotes the set of all potential paths from v_1 to v_i, then $X_i = \sum_{x \in \mathcal{P}_i} I(x)$ where $I(x)$ is a $\{0, 1\}$ indicator random variable indicating whether the path x exists in G; $I(x) = 1$ if and only if all edges in the path x are present in G. Then,

$$\begin{aligned}
\mathbb{P}(\mathcal{R}_i) = \mathbb{P}(X_i \geq 1) \leq \mathbb{E}[X_i] &= \sum_{x \in \mathcal{P}_i} \mathbb{E}[I(x)] \\
&= \sum_{\ell=0}^{i-2} \sum_{\substack{x \in \mathcal{P}_i \\ |x| = \ell+1}} \mathbb{E}[I(x)] \\
&= \sum_{\ell=0}^{i-2} \binom{i-2}{\ell} p^{\ell+1} = p(1+p)^{i-2} \leq p e^{pi}.
\end{aligned}$$

Let X denote the number of reachable vertices (other than v_1).

$$\mathbb{E}[X] = \sum_{i=2}^{n} \mathbb{P}(\mathcal{R}_i) \leq \sum_{i=1}^{n} p e^{pi} = p \cdot \frac{e^{p(n+1)} - 1}{e^p - 1}.$$

For $p = \frac{c \log n}{n} + \xi(n)$ with $c < 1$,

$$e^{p(n+1)} - 1 = \exp\left(c \log n + o(\log n)\right) - 1 = n^{c + o(1)},$$

and

$$\frac{p}{(e^p - 1)} = \left(\sum_{k=1}^{\infty} \frac{p^{k-1}}{k!} \right)^{-1} = 1 + O(p).$$

Thus,

$$\mathbb{E}[X] \le n^{c+o(1)}.$$

Applying Markov's inequality yields that $\mathbb{P}(X > \log(n)\mathbb{E}[X]) = o(1)$, so $X \le \log(n)\mathbb{E}[X] = n^{c+o(1)}$, a.a.s.

Now consider $c > 1$. Let

$$\begin{aligned} \xi'(n) &:= \frac{3}{c} \max\left\{ \frac{n}{\log\log n}, \frac{n^2 \xi(n)}{c \log n} \right\} \\ t &:= \frac{n}{c} - \xi'(n) \end{aligned}$$

Note that by our choice of $\xi'(n)$, and the fact that $\xi(n) = o(\frac{\log n}{n})$, that $\xi'(n) = o(n)$. Then,

$$\begin{aligned} \mathbb{P}(\mathcal{R}_t) \le pe^{pt} &= p \exp\left(\left(c\frac{\log n}{n} + \xi(n) \right) \left(\frac{n}{c} - \xi'(n) \right) \right) \\ &= p \exp\left(\log(n) + \frac{n\xi(n)}{c} - \frac{c(\log n)\xi'(n)}{n} - \xi(n)\xi'(n) \right) \\ &\le (1 + o(1))c \exp\left(\log\log(n) + \frac{n\xi(n)}{c} - \frac{c(\log n)\xi'(n)}{n} \right) = o(1) \end{aligned}$$

Here, the last inequality comes from the fact that, by our choice of $\xi'(n)$,

$$\frac{c(\log(n))\xi'(n)}{n} - \log\log(n) - \frac{n\xi(n)}{c} \ge \frac{1}{3}\xi'(n).$$

Since pe^{pi} is increasing in i, the expected number of reachable vertices v_i with $i \le t$ is at most $t\mathbb{P}(\mathcal{R}_t) = o(n)$. Applying Markov's inequality, $|\mathcal{R} \cap \{v_1, \ldots, v_t\}| = o(n)$ a.a.s. Thus,

$$|\mathcal{R}| \le n - t + |\mathcal{R} \cap \{v_1, \ldots, v_t\}| = \left(1 - \tfrac{1}{c} + o(1)\right) n \text{ a.a.s.}$$

For $p = \frac{\log n}{n} + \xi(n)$ with $\xi(n) = o\left(\frac{\log n}{n}\right)$, we will write $\xi(n) = \omega(n)\frac{\log n}{n}$, where $\omega(n) \to 0$. Let $t = n \cdot \left(1 - \omega(n) - \frac{1}{\log\log n}\right)$. Then,

$$\begin{aligned} \mathbb{P}(\mathcal{R}_t) \le pe^{-pt} &= \exp\left[(1 + \omega(n)) \left(1 - \omega(n) - \tfrac{1}{\log\log n}\right) \log n + \log\left((1 + \omega(n)) \tfrac{\log n}{n} \right) \right] \\ &= \exp\left[-\omega(n)^2 \log n - (1 + \omega(n)) \frac{\log n}{\log\log n} + \log\log n + \log(1 + \omega(n)) \right] \\ &= o(1), \end{aligned}$$

Thus the expected value of $|\mathcal{R} \cap \{v_1, \ldots, v_t\}|$ is $o(n)$ and by Markov's inequality, this is also true a.a.s. Now, since $n - t$ is also $o(n)$, we have that $R = o(n)$ a.a.s. $\square$

To prove the lower bounds, we require a simple lemma similar to Dirichlet's theorem. Let $d(i)$ denote the number of divisors of i and let $d_t(i)$ denote the number of divisors of i that are at most t. Dirichlet's Theorem states that

$$\sum_{i=1}^{k} d(i) = k \log k + (2\gamma - 1)k + O(\sqrt{k}),$$

where γ is Euler's constant. For our purposes, we need a refinement of this result, summing $d_t(i)$.

Lemma 1. $\sum_{i=1}^{k} d_t(i) = k \log \min(t, k) + O(k)$.

Proof. For $t > k$ the result follows from Dirichlet's theorem as we may replace $d_t(i)$ with $d(i)$ in the summation. For $t \leq k$,

$$\sum_{i=1}^{k} d_t(i) = k + \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \cdots + \left\lfloor \frac{k}{t} \right\rfloor \leq k\mathcal{H}_t,$$

where $\mathcal{H}_t$ is the t-th harmonic number. $\square$

Lower bounds: For exposition, assume that we construct our graph on countably many vertices and that we then restrict our attention to the first n vertices. Let X_i denote the index of the i-th reachable vertex (that is not v_1). If $X_i > n$ then $|\mathcal{R}| \leq i$. Set $X_0 = 1$, and for $i \geq 1$, $X_i - X_{i-1}$ is geometrically distributed with parameter $1 - (1-p)^i$. Fix t, and consider $\mathbb{E}[X_t]$:

$$\mathbb{E}[X_t] = \sum_{k=1}^{t} \mathbb{E}[X_k - X_{k-1}] = \sum_{k=1}^{t} \frac{1}{1 - (1-p)^k}.$$

Each term is an infinite geometric series, and so

$$\mathbb{E}[X_t] = \sum_{k=1}^{t} \sum_{j=0}^{\infty} (1-p)^{kj}.$$

As this series is absolutely summable (as $\mathbb{E}[X_t]$ is clearly finite), Fubini's theorem allows us to rearrange terms in the summation to get

$$\mathbb{E}[X_t] = t + \sum_{k=1}^{t} \sum_{j=1}^{\infty} (1-p)^{kj} = t + \sum_{i=1}^{\infty} d_t(i)(1-p)^i.$$

because the term $(1-p)^i$ appears in the original summation (where $i = kj$) once for every divisor i has that is at most t. We now use summation by parts to manipulate the second term:

$$\begin{aligned}
\sum_{i=1}^{\infty} d_t(i)(1-p)^i &= p \sum_{i=1}^{\infty} (1-p)^{i-1} \left(\sum_{\ell=1}^{i} d_t(\ell) \right) \\
&= p \sum_{i=1}^{\infty} (1-p)^{i-1} (i \log(\min\{t, i\}) + O(i)) \\
&\leq p(\log t + O(1)) \sum_{i=1}^{\infty} i(1-p)^{i-1}.
\end{aligned}$$

Since $\sum_{i=1}^{\infty} i(1-p)^{i-1} = 1/p^2$, we have that

$$\mathbb{E}[X_t] = t + \frac{\log t}{p} + O\left(\frac{1}{p}\right). \tag{1}$$

Furthermore, since $X_{k+1} - X_k$ and $X_k - X_{k-1}$ are independent,

$$\begin{aligned}\mathrm{Var}(X_t) &= \sum_{k=1}^{t} \frac{p}{(1-(1-p)^k)^2} \\ &\leq \sqrt{\left(\sum_{k=1}^{t} \frac{p^2}{(1-(1-p)^k)^3}\right)\left(\sum_{k=1}^{t} \frac{1}{(1-(1-p)^k)}\right)} \\ &\leq \sqrt{\frac{t}{p}\mathbb{E}[X_t]}. \end{aligned} \tag{2}$$

Here, the first inequality follows from an application of Cauchy-Schwarz, and the second from $\frac{p^2}{(1-(1-p)^k)^3} \leq \frac{p^2}{p^3} = \frac{1}{p}$.

Now, suppose that $p = c\frac{\log n}{n} + \xi(n)$ for $c < 1$, and set $t = n^c \exp(-n|\xi(n)| - \log\log(n))$. Then, from e̊q:Et,

$$\mathbb{E}[X_t] \leq n^c \exp(-n|\xi(n)|) + \frac{c\log n - 2n|\xi(n)| - \log\log n}{n^{-1}(c\log n + n\xi(n))} + O\left(\frac{\log n}{n}\right) \tag{3}$$

$$\leq n^c \exp(-n|\xi(n)|) + n - \frac{n^2|\xi(n)| - \log\log n}{(c\log n + n\xi(n))} + O\left(\frac{\log n}{n}\right) \tag{4}$$

$$= n\left(1 - \frac{n|\xi(n)| + \log\log(n)}{(c\log n + n\xi(n))} + o\left(\frac{n|\xi(n)| + \log\log n}{\log n}\right)\right). \tag{5}$$

For n sufficiently large, $\mathbb{E}[X_t] \leq n\left(1 - \frac{n|\xi(n)| + \log\log n}{2c\log n}\right)$. Meanwhile, from (2),

$$\mathrm{Var}(X_t) \leq (1+o(1))\sqrt{\frac{n^c}{\log n} \cdot (1+o(1))\frac{n}{c\log n} \cdot \mathbb{E}[X_t]} = \frac{n^{\frac{1}{2}(1+c)}}{\log n}\sqrt{\frac{\mathbb{E}[X_t]}{c}} = O\left(\frac{n^{3/2}}{\log n}\right),$$

because $\mathbb{E}[X_t] = O(n)$ and $c < 1$. Chebyshev's inequality asserts that

$$\mathbb{P}\left[|X_t - \mathbb{E}[X_t]| \geq \frac{n^2|\xi(n)| + n\log\log n}{2c\log n}\right] \leq \frac{4c^2\log^2 n \cdot \mathrm{Var}(X_t)}{(n^2|\xi(n)| + n(\log\log n))^2} = o(1).$$

Thus, $\mathbb{P}\left[X_t \leq \mathbb{E}[X_t] + \frac{n\log\log n}{2c\log n}\right] = 1 - o(1)$. Using e̊q:Et1,

$$\mathbb{P}\left[X_t \leq n\left(1 - \frac{\log\log n}{2c\log n} + o\left(\frac{\log\log n}{c\log n}\right)\right)\right] = 1 - o(1),$$

i.e., $X_t < n$ a.a.s. Since $X_t < n$ implies $|\mathcal{R}| \geq t$, we have that $|\mathcal{R}| > n^c \exp(-n|\xi(n)| - \log\log(n)) = n^{c+o(1)}$ a.a.s.

For $c > 1$, take $t = \frac{n\log\log n}{\log n}$. Then, using e̊q:Et,

$$\mathbb{E}[X_t] \leq \frac{n}{c} + o(n).$$

Again, by (2) and because $\mathbb{E}[X_t] = O(n)$, $\mathrm{Var}(X_t) = O(n^{3/2}\sqrt{\log\log n}/\log n) = o(n^{3/2})$. Chebyschev's inequality asserts that

$$\mathbb{P}\left[|X_t - \mathbb{E}[X_t]| \geq n^{3/4}\right] \leq \frac{o(n^{3/2})}{n^{3/2}} = o(1).$$

Hence,

$$\mathbb{P}\left[X_t \leq \mathbb{E}[X_t] + n^{3/4}\right] = 1 - o(1). \tag{6}$$

So, $X_t \leq \frac{n}{c} + o(n)$ a.a.s. We now consider the vertices indexed higher than X_t and show that essentially all of them are reachable. Let Y be the vertices with index higher than X_t which are *not* adjacent to one of the first t reachable vertices in $v_1, \ldots, v_{X_t}$. Then

$$\mathbb{E}[|Y|] = \sum_{j=X_t+1}^{n} (1-p)^t = (n - X_t)(1-p)^t \leq ne^{-pt} = \frac{n}{\log^{c+o(1)} n} = o(n).$$

Applying Markov's inequality, $|Y| = o(n)$ with probability $1-o(1)$. Since the set of vertices indexed above X_t that is not reachable is a subset of Y, $|\mathcal{R}| \geq t + (n - X_t) - |Y|$. Since $|Y|, t$ are $o(n)$ and $X_t = \frac{n}{c} + o(n)$, we have that $|\mathcal{R}| \geq n(1 - \frac{1}{c} + o(1))$ with probability $1 - o(1)$, as desired. $\square$

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